

**Correlation of inflation-produced magnetic fields with scalar fluctuations**

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(Received 27 September 2011; published 29 December 2011)

If the conformal invariance of electromagnetism is broken during inflation, then primordial magnetic fields may be produced. If this symmetry breaking is generated by the coupling between electromagnetism and a scalar field—e.g. the inflaton, curvaton, or Ricci scalar—then these magnetic fields may be correlated with primordial density perturbations, opening a new window to the study of non-Gaussianity in cosmology. In order to illustrate, we couple electromagnetism to an auxiliary scalar field in a de Sitter background. We calculate the power spectra for scalar-field perturbations and magnetic fields, showing how a scale-free magnetic-field spectrum with rms amplitude of  $\sim \text{nG}$  at Mpc scales may be achieved. We explore the Fourier-space dependence of the cross correlation between the scalar field and magnetic fields, showing that the dimensionless amplitude, measured in units of the power spectra, can grow as large as  $\sim 500H_I/M$ , where  $H_I$  is the inflationary Hubble constant and  $M$  is the effective mass scale of the coupling.

DOI: [10.1103/PhysRevD.84.123525](https://doi.org/10.1103/PhysRevD.84.123525)

PACS numbers: 98.80.-k, 98.62.En, 98.65.-r, 98.80.Cq

**I. INTRODUCTION**

The predictions of the simplest single-field slow-roll models of inflation agree remarkably well with current cosmological data, yet experience gained from effective field theories suggests that this model is likely not the whole story. A vast literature has now arisen to explore ultraviolet completions and their predictions for future, more sensitive, observations [1,2]. One of the principle lines of investigation has been the predictions for non-Gaussianity due to self-couplings, nontrivial inflaton kinetic terms, or interactions between multiple fields associated with inflation [3–5].

Another possibility for beyond single-field slow-roll physics is coupling of the inflaton, or some other spectator field, to electromagnetism. If such a coupling breaks the conformal invariance of electromagnetism, then quantum fluctuations in the electromagnetic field may be amplified into classical magnetic fields in much the same way as quantum fluctuations in the inflaton (graviton) become density perturbations (gravitational waves). It has been suggested that such inflation-produced magnetic fields may provide the seed fields required for galactic dynamos [6–14], but it may also be that the signatures of such magnetic fields may be observed in the cosmic microwave background [15–28], and thus shed light on inflation, even if they are unrelated to galactic magnetism. Either way, the search for primordial magnetic fields provides an additional observational probe of the physics of inflation to parallel that obtained from non-Gaussianity searches.

Here we explore the cross correlation between primordial magnetic fields and a scalar field in a toy model in which the scalar field is coupled to electromagnetism, with no gravity, in a fixed de Sitter background. The homogeneous time evolution of the scalar field breaks the conformal invariance of electromagnetism. We first calculate the quantum mechanical spectrum of scalar- and magnetic-field fluctuations produced, and we then calculate the cross correlation between the scalar and magnetic fields.

If the scalar field is a curvaton field, and if that curvaton is responsible for primordial perturbations, then the scalar-field-magnetic-field cross correlation we calculate will be precisely the density-magnetic-field correlation observed in the Universe today. Our calculation also illustrates the principal ingredients that will arise in a density-perturbation-magnetic-field correlation if the scalar field is the inflaton.

In Sec. II we introduce our model, work out the dynamical behavior, and evaluate the two-point statistics of the scalar and magnetic fields. In Sec. III we present the calculation of the cross correlation, and we analyze its behavior in Sec. IV. We conclude in Sec. V. Throughout, we work in spatially flat Robertson-Walker coordinates, with line element  $ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)$ .

**II. MECHANISM OF MAGNETIC-FIELD AMPLIFICATION**

The action for our model is

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} W(\phi) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right),$$

where  $\phi(\vec{x}, t)$  is the scalar field, and  $F_{\mu\nu}$  the electromagnetic field-strength tensor. The scalar-field potential is

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$V(\phi) = -3nMH_I^2\phi$ , and the coupling function is  $W(\phi) = e^{2\phi/M}$ . We suppose that some other field is driving inflation. In practice, we consider a fixed de Sitter background with Hubble constant  $H_I$ , whereby the scale factor is  $a(\eta) = -1/(\eta H_I)$  for the run of conformal times  $-\infty < \eta \leq \eta_I < 0$  and  $\eta_I$  marks the end of inflation. Aspects of this model have previously been studied [9,12], but we revisit the details in preparation for our later calculations.

### A. Scalar field

The scalar-field equation of motion is

$$\square\phi = \frac{\partial V}{\partial\phi} + \frac{1}{4} \frac{\partial W}{\partial\phi} F_{\mu\nu} F^{\mu\nu}, \quad (1)$$

and it has a solution

$$\phi = c_0 + c_1 \eta^3 - nM \ln(\eta/\eta_I), \quad (2)$$

where we assume there is no homogeneous electric or magnetic field. We take the integration constants  $c_0$  and  $c_1$  to vanish so that  $W(\phi) = 1$  at the end of inflation. In this way, the usual electromagnetic Lagrangian is recovered for the post-inflationary epoch, and we assume that some mechanism stops the subsequent evolution of  $\phi$ , so that the standard Maxwell equations are preserved at all times after inflation. We also define  $I(\eta) \equiv [W(\phi(\eta))]^{1/2} = (\eta/\eta_I)^{-n}$ , which will appear in our analysis of the electromagnetic field.

The scalar field has fluctuations  $\delta\phi(\vec{x}, \eta)$ , about its homogeneous component, described by the evolution equation,

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + (a^2 V_{\phi\phi} - \nabla^2)\delta\phi = 0, \quad (3)$$

where  $\mathcal{H} = a'/a$  and  $\nabla^2$  is the spatial Laplace operator. To be clear, we fix the background to be pure de Sitter spacetime and subsequently “turn off” gravity, so that there are no fluctuations of the spacetime metric. Following standard procedures, the quantized scalar field is decomposed in terms of time-dependent mode functions  $\delta\phi_k$ ,

$$\delta\phi(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} [e^{i\vec{k}\cdot\vec{x}} \delta\phi_k(\eta) \hat{a}_k + \text{H.c.}], \quad (4)$$

where  $\hat{a}_k$  and  $\hat{a}_k^\dagger$  are, respectively, annihilation and creation operators that satisfy  $[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3 \delta(\vec{k} - \vec{k}')$ . The uncertainty relation for the scalar field and its conjugate momentum  $\delta\phi'$ ,

$$[\delta\phi(\vec{x}, \eta), \delta\phi'(\vec{y}, \eta)] = i\delta(\vec{x} - \vec{y})/a^2(\eta), \quad (5)$$

results in a constraint on the two linearly independent solutions to the mode equation. Because the effective mass is zero,  $V_{\phi\phi} = 0$ , we obtain the solution

$$\delta\phi_k(\eta) = \frac{H_I}{\sqrt{2}} \frac{i - k\eta}{k^{3/2}} e^{-ik\eta}, \quad (6)$$

corresponding to the Bunch-Davies state, having positive frequency in the remote past,  $\eta \rightarrow -\infty$  for  $k|\eta| \gg 1$ . The requirement  $\delta\phi \ll \phi$  that the fluctuations are small translates into the bound  $H_I/M \ll 1$ . Finally, the two-point correlation function is

$$\langle \delta\phi(\vec{x}, \eta) \delta\phi(\vec{y}, \eta) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} P_{\delta\phi}(k), \quad (7)$$

where the scalar-field power spectrum—defined by  $\langle \delta\phi_{\vec{k}} \delta\phi_{\vec{k}'}^* \rangle = (2\pi)^3 \delta_D(\vec{k} - \vec{k}') P_{\delta\phi}(k)$  and  $\delta_D$  is the Dirac delta function—is  $P_{\delta\phi}(k) = H_I^2/2k^3$ , valid for modes outside the horizon at the end of inflation.

The root-mean-squared amplitude—the correlation function at zero lag (at  $\vec{x} = \vec{y}$ )—is divergent at both the infrared and ultraviolet limits. Hence, we bound the run of wave numbers to  $[k_{\min}, k_{\max}]$ , so that

$$\delta\phi_{\text{rms}} \equiv \langle (\delta\phi)^2 \rangle^{1/2} = \frac{H_I}{2\pi} (\ln k_{\max}/k_{\min})^{1/2} \quad (8)$$

gives the rms scalar-field fluctuation. In practice, we associate the minimum wave number with the present-day Hubble radius—i.e.,  $k_{\min} = 2\pi H_0$ —and the maximum wave number with an astrophysical scale that we indicate by  $\lambda$ .

### B. Electromagnetism

The full action for electromagnetism includes not only the free Maxwell field, but also the coupling to charged particles as well as the action for the charged particles themselves. Including these additional terms, we may write

$$S_{\text{em}} = - \int d^4x \sqrt{-g} \left[ \frac{1}{4} I^2(\phi) F_{\mu\nu} F^{\mu\nu} + A^\mu J_\mu + \mathcal{L}_q \right], \quad (9)$$

where  $\mathcal{L}_q$  is the Lagrangian for charged particles. The electromagnetic coupling, or electric charge, is inversely proportional to  $I(\eta)$ . Consequently, in the case  $n > 0$  the coupling is strong at early times [12]. Such a strong-coupling scenario has previously been dismissed [12], since the free-field behavior of electromagnetic waves would no longer be valid. We therefore consider here the alternative Lagrangian,

$$S_{\text{em}} = - \int d^4x \sqrt{-g} I^2(\phi) \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A^\mu J_\mu + \mathcal{L}_q \right], \quad (10)$$

in which the conformal factor  $I^2(\phi)$  is moved outside the entire electromagnetic-sector Lagrangian. With this modification, the strong-coupling problem is alleviated. This gauge-invariance violating Lagrangian could arise in reduction from a higher dimensional theory, although in that case we might not expect the same conformal factor  $I^2(\phi)$  to also multiply the mass term of the charged particle.

The effect of the coupling function on Maxwell's equations is straightforward. In the absence of charges or currents, only Ampere's equation is modified, to

$$\vec{\nabla} \times \vec{B} = \frac{1}{a^2 W} \frac{\partial}{\partial \eta} (a^2 W \vec{E}), \quad (11)$$

where we have assumed  $W$  is solely a function of (conformal) time. Faraday's law remains unaltered:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{a^2} \frac{\partial}{\partial \eta} (a^2 \vec{B}). \quad (12)$$

The curl in each of the above two equations vanishes for homogeneous fields, implying that  $|\vec{B}| \propto a^{-2} \propto \eta^2$  and  $|\vec{E}| \propto W^{-1} a^{-2} \propto \eta^{2+2n}$ . The magnetic- and electric-field energy densities therefore scale as  $\rho_B = WB^2/8\pi \propto \eta^{4-2n}$  and  $\rho_E = WE^2/8\pi \propto \eta^{4+2n}$ . Recalling that the conformal time runs from a large and negative value at the beginning of inflation to a small and negative value close to zero at the end of inflation, then for  $n = 2$  and  $n = -3$  (special cases we will consider below), the magnetic-field energy density remains constant or decays, respectively. The electric-field energy density decays for  $n = 2$ , but it grows, as  $\rho_E \propto \eta^{-2}$ , for  $n = -3$ . In this latter case, the energy density in the electric-field component of the quantum-mechanically induced electromagnetic fields will, if inflation goes on long enough, ultimately dominate the energy density,  $\sim 3H_I^2/(8\pi G)$ , in the inflaton. As we will see below (see also Ref. [12]), this then severely restricts the number of  $e$ -foldings of inflation. We will thus ultimately discard the  $n = -3$  case.

### C. Quantum fluctuations of the magnetic field

The action for the free-field theory is

$$S_{\text{em}} = - \int d^4x \sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ = \int d\eta d^3x [I(\eta)]^2 \left( \frac{1}{2} A_i^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \right), \quad (13)$$

in the Coulomb gauge, where  $A_i$  is the vector potential. The Latin indices here are contracted using the spatial part of the Minkowski metric. Defining the vector field  $V_i = I(\eta) A_i$  we can bring the kinetic term to canonical form, whereby

$$S_{\text{em}} = \int d\eta d^3x \frac{1}{2} \left[ V_i'^2 - (\partial_i V_j)^2 + \frac{I''}{I} V_i^2 \right], \quad (14)$$

after some integrations by parts. The quantized field  $V_i$  is expanded in terms of time-dependent mode functions  $v_k(\eta)$ ,

$$V_i(\vec{k}, \eta) = \sum_{\sigma=1}^2 \int \frac{d^3k}{(2\pi)^3} [e^{i\vec{k}\cdot\vec{x}} v_k(\eta) e_i^{(\sigma)}(\hat{k}) \hat{b}_\sigma(k) + \text{H.c.}], \quad (15)$$

where  $\hat{b}$ ,  $\hat{b}^\dagger$  are annihilation and creation operators satisfying  $[\hat{b}_\sigma(\vec{k}), \hat{b}_\sigma^\dagger(\vec{k}')] = (2\pi)^3 \delta_{\sigma,\sigma'} \delta_D(\vec{k} - \vec{k}')$ , where  $e_i^{(\sigma)}$  is the polarization vector,  $\sigma$  sums over the two linear-polarization

states, and  $\sum_\sigma e_i^{(\sigma)}(\hat{k}) e_j^{(\sigma)}(\hat{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ , which further ensures transversality as a consequence of the gauge choice. Canonical quantization means that the vector field and its conjugate momentum  $V_i'$  satisfy the commutation relation,

$$[V_i(\vec{x}, \eta), V_j'(\vec{y}, \eta)] = i\delta_{ij} \delta(\vec{x} - \vec{y}), \quad (16)$$

which results in a constraint on the two linearly independent solutions to the mode equation. The scalar field contributes an effective time-dependent mass term to the vector field, so that the mode functions obey the equation

$$v_k'' + \left( k^2 - \frac{I''}{I} \right) v_k = 0, \quad (17)$$

where  $I''/I = n(n+1)/\eta^2$  is positive for  $n > 0$  or  $n' = n+1 < 0$ . At high frequencies,  $k|\eta| \gg 1$ , the solutions are oscillatory, but at low frequencies the scalar field causes solutions to grow as  $v_k \propto |\eta|^{-n}$ ,  $|\eta|^{1+n}$ . The normalized solution, having positive frequency in the remote past,  $\eta \rightarrow -\infty$ , for  $k|\eta| \gg 1$ , is

$$v_k(\eta) = \sqrt{\frac{\pi}{2}} \frac{(-k\eta)^{1/2}}{\sqrt{2k}} e^{i\pi(1+n)/2} H_{(1/2)+n}^{(1)}(-k\eta), \quad (18)$$

where  $H_n(x)$  is a Hankel function. In this case, the two-point correlation function for the magnetic field is

$$\langle \vec{B}(\vec{x}, \eta) \cdot \vec{B}(\vec{y}, \eta) \rangle = \frac{1}{a(\eta)^4} \left( \delta_{ij} \frac{\partial^2}{\partial x^k \partial y^k} - \frac{\partial^2}{\partial x^j \partial y^i} \right) \\ \times \langle A_i(\vec{x}, \eta) A_j(\vec{y}, \eta) \rangle \\ = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} P_B(k), \quad (19)$$

where

$$P_B(k) = \frac{\pi}{2} \frac{H_I^4}{k^3} \left( \frac{\eta}{\eta_I} \right)^{2n} \\ \times (-k\eta)^5 H_{(1/2)+n}^{(1)}(-k\eta) H_{(1/2)+n}^{(2)}(-k\eta) \quad (20)$$

is the magnetic-field power spectrum. In the unamplified case, corresponding to  $n = 0$ , we have  $P_B^{(0)} = k(H_I \eta_I)^4$  at the end of inflation; the correlations in this case are then the usual vacuum-fluctuation correlations. Production of classical long-wavelength magnetic fields occurs for  $n > 0$  or for  $n' = n+1 < 0$ . To treat both cases with a single expression, we define  $n_B = 4 - 2n$  for the case  $n \geq 0$  and  $n_B = 4 + 2n'$  for  $n' = n+1 < 0$ . Consequently, the power spectrum is

$$P_B \simeq \frac{\Gamma(\frac{5-n_B}{2})^2}{\pi} (-k\eta_I/2)^{n_B-4} P_B^{(0)}, \quad (21)$$

for modes outside the horizon at the end of inflation. Since  $k|\eta_I| \ll 1$  for modes outside the horizon at the end of inflation, and since  $n_B - 4 < 0$ , the amplified ratio  $P_B/P_B^{(0)}$  can grow quite large.

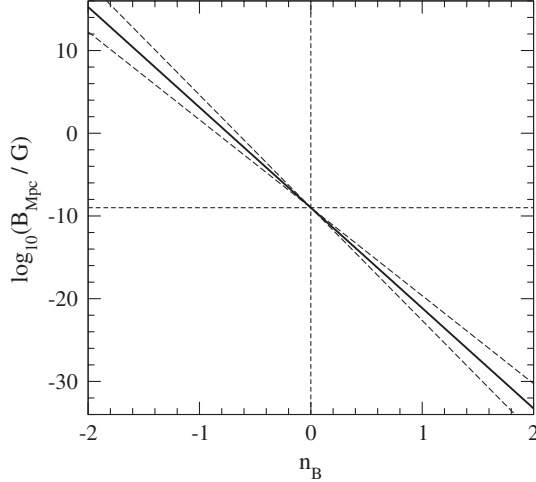


FIG. 1. The magnetic field strength  $B_{\text{Mpc}} = (d\langle B^2 \rangle / d \ln k)^{1/2}$  at 1 Mpc is shown by the solid line as a function of the index  $n_B$ . Horizontal and vertical short-dashed lines indicate the nG field strength obtained for  $n_B = 0$ . The effect of the  $\Gamma$  function in Eq. (23) is negligible compared to the exponential factor  $10^{-18-24.3n_B}$ . The long-dashed lines rotated clockwise and counterclockwise show the magnetic field strength at Gpc and kpc scales, respectively.

The mean-squared magnetic-field power in long-wavelength modes at the end of inflation, per logarithmic interval, is

$$\frac{d}{d \ln k} \langle B^2 \rangle \simeq \left(\frac{2}{\pi}\right)^3 \Gamma\left(\frac{5-n_B}{2}\right)^2 H_I^4 (-k\eta_I/2)^{n_B}. \quad (22)$$

A scale-free spectrum  $n_B = 0$  can be achieved for  $n = 2, -3$ . Using  $(\text{Gauss})^2/8\pi = 1.91 \times 10^{-40} \text{ GeV}^4$ ,  $\text{Mpc} = 1.56 \times 10^{38} \text{ GeV}^{-1}$ , estimating  $|\eta_I| \sim 10^{-27} \text{ Gpc}$  (consistent with  $H_I \simeq 10^{14} \text{ GeV}$  and  $z_I \simeq 10^{28}$  for the redshift to the end of inflation), and then redshifting to the present day, we find

$$\frac{d}{d \ln k} \langle B^2 \rangle \simeq 10^{-18-24.3n_B} \frac{\Gamma(\frac{5-n_B}{2})^2}{\Gamma(5/2)^2} \left(\frac{k}{\text{Mpc}^{-1}}\right)^{n_B} \text{ G}^2. \quad (23)$$

If  $n_B = 0$  or  $n = 2$  or  $-3$ , then the field strength is roughly  $10^{-9} \text{ G}$ , which may be sufficient to explain the observed astrophysical and cosmological magnetic fields [8]. The dependence of the field strength at 1 Mpc as a function of the index  $n_B$  is shown in Fig. 1.

#### D. Energy density of the magnetic and electric fluctuations

The same magnetic-field spectrum is obtained for two values of the index  $n$ . However, the time evolution of the coupling function  $I(\eta)$  breaks the usual duality between electric and magnetic fields, and the electric-field energy density may in some cases increase, as discussed above. We require the energy density in superhorizon modes of the electromagnetic fields to be smaller than the energy

density of the inflaton, and thereby derive now a restriction on the allowed values of  $n$  and  $H_I$ .

The stress-energy tensor that appears as a source for the Einstein equations is

$$T^{\mu\nu} = I^2(\phi)(g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}). \quad (24)$$

The energy density observed in the cosmic rest frame is

$$\begin{aligned} \rho_{EB} &= \frac{I^2(\eta)}{2a^4(\eta)} \langle A'_i A'_i + (\partial_i A_j)(\partial_i A_j) - (\partial_i A_j)(\partial_j A_i) \rangle \\ &= \lim_{\vec{x} \rightarrow \vec{y}} \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} P_\rho(k), \end{aligned} \quad (25)$$

where the final term in the top line ultimately vanishes due to the transversality condition  $\hat{k} \cdot \vec{e}$ . The energy-density power spectrum consists of two terms, a kinetic term due to the electric field and a spatial-gradient term due to the magnetic field,

$$P_\rho(k) = \frac{3(2\pi)^3}{2a^4(\eta)} \left( \left| I(\eta) \left( \frac{v_k(\eta)}{I(\eta)} \right)' \right|^2 + k^2 |v_k(\eta)|^2 \right). \quad (26)$$

The integral over wave numbers runs from  $k_{\min} = -1/\eta_S$  to  $k_{\max} = -1/\eta$ , where  $\eta_S = \eta_I e^{N_I}$  is the conformal time at the beginning of  $N_I$   $e$ -foldings of inflation, thereby spanning the range of wavelengths that have exited the horizon by the time  $\eta$ . The pattern of behavior distinguishes two regimes,

$$\rho_{EB} = H_I^4 \times \begin{cases} \mathcal{O}(1) & \text{for } |n| \leq 2 \\ \mathcal{O}(1) \times \left(\frac{\eta_S}{\eta}\right)^{2(|n|-1)} & \text{for } |n| > 2. \end{cases} \quad (27)$$

In the first case, which includes the scale-free solution  $n = 2$ , the energy density is simply proportional to  $H_I^4$  which is always subdominant to the inflaton energy density. However, the second case, which includes the other scale-free solution  $n = -3$ , places severe restrictions,

$$|n| < 2 + \frac{1}{N_I} \ln \frac{M_P}{H_I}, \quad (28)$$

on the index  $n$ . Since observational constraints limit  $H_I \lesssim 10^{-5} M_P$ , then to achieve at least 60  $e$ -foldings of inflation, the index is bounded by  $|n| < 2.2$ , thereby eliminating the case  $n = -3$ . At the value  $n = -2.2$ , Eq. (23) tells us that the magnetic-field strength on Mpc scales is roughly  $10^{-30} \text{ G}$ . The case  $n = 2$ , however, safely satisfies the above bound and yields a nG magnetic field as we have shown.

### III. CORRELATION OF MAGNETIC FIELDS AND SCALAR FLUCTUATIONS

We now evaluate the  $(\delta\phi)BB$  correlation making use of the in-in formalism [29]. After splitting the Hamiltonian into a free part plus an interaction part  $\hat{H}_{\text{int}}$ , we may evaluate, to first order in perturbation theory,



$$\begin{aligned} & \left\langle \frac{\delta\phi}{M}(\vec{x}, \eta) A_i(\vec{y}, \eta) A_j(\vec{z}, \eta) \right\rangle \\ &= - \int_{-\infty}^{\eta_I} d\eta_1 2\text{Im} \left[ \left\langle \hat{H}_{\text{int}}(\eta_1) \frac{\delta\phi}{M}(\vec{x}, \eta) A_i(\vec{y}, \eta) A_j(\vec{z}, \eta) \right\rangle \right]. \end{aligned} \quad (29)$$

The interaction Hamiltonian is

$$\hat{H}_{\text{int}} = - \int d^3x \left( \frac{\eta}{\eta_I} \right)^{-2n} \frac{\delta\phi}{M} \left( (A_i')^2 - \frac{1}{2} (\partial_i A_j - \partial_j A_i)^2 \right). \quad (30)$$

Using Eqs. (4) and (15), we find that the expectation value on the right-hand side of Eq. (29) is

$$\begin{aligned} & \left\langle \hat{H}_{\text{int}}(\eta') \frac{\delta\phi}{M}(\vec{x}, \eta_I) A_i(\vec{y}, \eta_I) A_j(\vec{z}, \eta_I) \right\rangle \\ &= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \\ & \quad \times e^{i\vec{k}_1 \cdot \vec{x} + i\vec{k}_2 \cdot \vec{y} + i\vec{k}_3 \cdot \vec{z}} (K_{ij}^{(1)} + K_{ij}^{(2)}). \end{aligned} \quad (31)$$

The functions  $K_{ij}^{(1)}$  and  $K_{ij}^{(2)}$  are defined as

$$\begin{aligned} K_{ij}^{(1)} &= -\frac{2}{M^2} \delta_{ij} \left( \frac{\eta'}{\eta_I} \right)^{-2n} \delta\phi_{k_1}(\eta') \delta\phi_{k_1}^*(\eta_I) \\ & \quad \times \left( \frac{d}{d\eta'} A_{k_2}(\eta') \right) A_{k_2}^*(\eta_I) \left( \frac{d}{d\eta'} A_{k_3}(\eta') \right) A_{k_3}^*(\eta_I), \end{aligned} \quad (32)$$

$$\begin{aligned} K_{ij}^{(2)} &= -\frac{2}{M^2} (\vec{k}_2 \cdot \vec{k}_3 \delta_{ij} - k_{2j} k_{3i}) \left( \frac{\eta'}{\eta_I} \right)^{-2n} \delta\phi_{k_1}(\eta') \delta\phi_{k_1}^*(\eta_I) \\ & \quad \times A_{k_2}(\eta') A_{k_2}^*(\eta_I) A_{k_3}(\eta') A_{k_3}^*(\eta_I), \end{aligned} \quad (33)$$

where we indicate the scalar mode functions of the vector potential as  $A_k(\eta) = v_k(\eta)/I(\eta)$ . Plugging Eqs. (32) and (33) into Eq. (31), we find

$$\begin{aligned} & \left\langle \frac{\delta\phi}{M}(\vec{x}, \eta_I) A_i(\vec{y}, \eta_I) A_j(\vec{z}, \eta_I) \right\rangle \\ &= \int \prod_{i=1}^3 \frac{d^3k_i}{(2\pi)^3} e^{i(\vec{k}_1 \cdot \vec{x} + \vec{k}_2 \cdot \vec{y} + \vec{k}_3 \cdot \vec{z})} (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) U_{ij}, \end{aligned} \quad (34)$$

$$\begin{aligned} U_{ij} &= -2\text{Im} \int d\eta' (K_{ij}^{(1)} + K_{ij}^{(2)}) \\ &= -\frac{\pi^2}{8} \left( \frac{H_I}{M} \right)^2 \frac{1}{k_1^4} (\delta_{ij} I_1 + (\hat{k}_2 \cdot \hat{k}_3 \delta_{ij} - \hat{k}_{2j} \hat{k}_{3i}) I_2), \end{aligned} \quad (35)$$

where we introduce the integrals

$$\begin{aligned} I_1 &= \text{Im} \int_1^\infty d\mu u_1 (i + \mu u_1) (-i + u_1) e^{iu_1(\mu-1)} \mu^{-2n} \\ & \quad \times \frac{d}{d\mu} [\mu^{(1/2)+n} H_{(1/2)+n}^{(1)}(\mu u_2)] H_{(1/2)+n}^{(2)}(u_2) \\ & \quad \times \frac{d}{d\mu} [\mu^{(1/2)+n} H_{(1/2)+n}^{(1)}(\mu u_3)] H_{(1/2)+n}^{(2)}(u_3), \end{aligned} \quad (36)$$

$$\begin{aligned} I_2 &= \text{Im} \int_1^\infty d\mu u_1 (i + \mu u_1) (-i + u_1) e^{iu_1(\mu-1)} \\ & \quad \times u_2 u_3 \mu H_{(1/2)+n}^{(1)}(\mu u_2) H_{(1/2)+n}^{(2)}(u_2) \\ & \quad \times H_{(1/2)+n}^{(1)}(\mu u_3) H_{(1/2)+n}^{(2)}(u_3). \end{aligned} \quad (37)$$

While  $I_2$  and the magnetic-field power spectrum are both invariant under  $n \rightarrow 1 + n$ ,  $I_1$  is not. This is not surprising since the interaction Hamiltonian is not invariant under this operation. In the above, we have defined  $\mu = \eta/\eta_I$  and  $u_i = -k_i \eta_I$  for  $i = 1, 2, 3$ .

The three-point correlation function for the scalar field with the magnetic field is obtained from

$$\begin{aligned} & \left\langle \frac{\delta\phi}{M}(\vec{x}, \eta_I) \vec{B}(\vec{y}, \eta_I) \cdot \vec{B}(\vec{z}, \eta_I) \right\rangle \\ &= -2\text{Im} \int_{-\infty}^{\eta_I} d\eta \left\langle H_{\text{int}}(\eta) \frac{\delta\phi}{M}(\vec{x}, \eta_I) \vec{B}(\vec{y}, \eta_I) \cdot \vec{B}(\vec{z}, \eta_I) \right\rangle \end{aligned} \quad (38)$$

$$= \frac{1}{a(\eta_I)^4} \left( \delta_{ij} \frac{\partial^2}{\partial y^k \partial z^k} - \frac{\partial^2}{\partial y^j \partial z^i} \right) \left\langle \frac{\delta\phi}{M}(\vec{x}, \eta) A_i(\vec{y}, \eta) A_j(\vec{z}, \eta) \right\rangle. \quad (39)$$

After some calculations, the final result is

$$\begin{aligned} & \left\langle \frac{\delta\phi}{M}(\vec{x}, \eta_I) \vec{B}(\vec{y}, \eta_I) \cdot \vec{B}(\vec{z}, \eta_I) \right\rangle \\ &= \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} e^{i(\vec{k}_1 \cdot \vec{x} + \vec{k}_2 \cdot \vec{y} + \vec{k}_3 \cdot \vec{z})} (2\pi)^3 \\ & \quad \times \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) P_3(k_1, k_2, k_3), \end{aligned} \quad (40)$$

where

$$\begin{aligned} P_3(k_1, k_2, k_3) &= \frac{\pi^2}{8} \left( \frac{H_I}{M} \right)^2 \frac{1}{a(\eta_I)^4} \frac{k_2 k_3}{k_1^4} \\ & \quad \times (2\hat{k}_2 \cdot \hat{k}_3 I_1 + (1 + (\hat{k}_2 \cdot \hat{k}_3)^2) I_2). \end{aligned} \quad (41)$$

From statistical isotropy, the function  $P_3(k_1, k_2, k_3)$  depends only on the magnitudes of the three wave vectors, and we have used  $\hat{k}_2 \cdot \hat{k}_3 = (k_1^2 - k_2^2 - k_3^2)/(2k_2 k_3)$  in Eq. (41). Equations (40) and (41), along with Eqs. (36) and (37), form the main results on which our subsequent analysis is based.

#### IV. ANALYSIS OF CROSS CORRELATION

We would like to analyze the cross correlation between the primordial magnetic field and the scalar field to determine if there is any imprint or unique signature that would indicate the scalar field's role in the amplification.

##### A. The amplified cross-correlation power spectrum

To start, we calculate the cross-correlation power spectrum for several trial cases. The integrals  $I_{1,2}$  can be evaluated analytically for integer values of  $n$ . In most cases, the results are cumbersome, so we assume  $u_i \ll 1$  after carrying out the integrals in order to shorten the expressions. For example, for  $n = 0$ ,

$$I_1 = -I_2 = -\frac{4}{\pi^2} \frac{u_1(u_1 + \omega)}{\omega^2}, \quad (42)$$

where  $\omega = u_1 + u_2 + u_3$ . Plugging in these results, we find

$$P_3(k_1, k_2, k_3) = \frac{(H_I/M)^2 (2k_1 + k_2 + k_3)(k_1 - k_2 - k_3)^2}{a(\eta_I)^4 8k_1^3 k_2 k_3}, \quad (43)$$

where we have used  $\cos\theta = \hat{k}_2 \cdot \hat{k}_3 = (k_1^2 - k_2^2 - k_3^2)/(2k_2 k_3)$  for the angle between the vectors  $k_2$  and  $k_3$ .

Amplification occurs for  $n > 0$  and  $n < -1$ , so that, for comparison, we consider integer cases  $n = 1$  and  $n = -2$  whereupon the integration simplifies. For  $n = 1$ , we find

$$I_1 = \frac{4}{\pi^2} \frac{u_1(u_1 + \omega)}{u_2 u_3 \omega^2}, \quad (44)$$

$$I_2 = \frac{4}{\pi^2} \frac{u_1(\omega^3 + u_1^2 \omega - u_1 \omega^2 - u_2 u_3 \omega - u_1 u_2 u_3)}{u_2^2 u_3^2 \omega^2}. \quad (45)$$

For  $n = -2$ ,

$$I_1 = -\frac{4}{\pi^2} \frac{u_1}{u_2^3 u_3^3 \omega^2} (3u_1^3 \omega^2 (\gamma + \ln \omega) + 3u_1^3 u_2 u_3 + 3u_1^2 (u_2^2 u_3 + u_2 u_3^2 - \omega^3) - u_2^2 u_3^2 (\omega + u_1)), \quad (46)$$

$$I_2 = \frac{4}{\pi^2} \frac{u_1(\omega^3 + u_1^2 \omega - u_1 \omega^2 - u_2 u_3 \omega - u_1 u_2 u_3)}{u_2^2 u_3^2 \omega^2}. \quad (47)$$

In the case of most interest,  $n = 2$ , the integrals yield

$$I_1 \simeq \frac{36}{\pi^2} \frac{u_1}{u_2^3 u_3^3 \omega^2} (\omega^3 - u_1 u_2 u_3 - \omega(u_1 u_2 + u_1 u_3 + u_2 u_3)), \quad (48)$$

$$I_2 \simeq \frac{36}{\pi^2} \frac{u_1}{u_2^4 u_3^4 \omega^2} (-3u_1^3 \omega^2 (\gamma + \ln \omega) + \omega^5 - 3u_1 \omega^4 + 3(2u_1^2 - u_2 u_3) \omega^3 + (3u_1 u_2 u_3 - u_1^3) \omega^2 + (u_2^2 u_3^2 - 3u_1^2 u_2 u_3) \omega + u_1 u_2^2 u_3^2). \quad (49)$$

Since  $|k\eta_I| \ll 1$  we have discarded subdominant terms from the above results. We note that the  $\ln \omega$  term above results in a large numerical coefficient, since  $-k\eta_I \sim 10^{-27}$  for modes that are just entering the horizon today. These expressions are inserted into Eq. (41) to find the cross-correlation power spectrum.

##### B. The real-space cross-correlation coefficient

Our next step is to determine the dimensionless magnitude of the cross correlation; i.e., how strongly does the magnetic-field energy density correlate with the scalar-field perturbation? We thus now calculate the zero-lag cross correlation  $\langle (\delta\phi)B^2 \rangle$  in units of  $\langle (\delta\phi)^2 \rangle^{1/2} \langle B^2 \rangle$ .

This cross-correlation amplitude, evaluated in the coincidence limit, can be evaluated as follows. Starting from Eqs. (40) and (41), we evaluate the  $\vec{k}_1$  integration to eliminate the delta function. The remaining integrand depends only on the magnitudes  $k_2$ ,  $k_3$ , and  $\theta$ , the angle between the two vectors:

$$\langle (\delta\phi)B^2 \rangle = \frac{M}{8\pi^4} \int k_2^2 dk_2 k_3^2 dk_3 d(\cos\theta) P_3(k_1, k_2, k_3), \quad (50)$$

where  $k_1 = (k_2^2 + k_3^2 + 2k_2 k_3 \cos\theta)^{1/2}$ . However, we can replace the  $\theta$  integral by  $k_1$ , whereby

$$\langle (\delta\phi)B^2 \rangle = \frac{M}{8\pi^4} \int k_2 dk_2 k_3 dk_3 \int_{|k_2 - k_3|}^{k_2 + k_3} k_1 dk_1 P_3(k_1, k_2, k_3). \quad (51)$$

Since the integrand is invariant under the exchange of  $k_2$  and  $k_3$ , we can replace  $P_3 \rightarrow 2P_3\theta(k_2 - k_3)$  and remove the absolute-value sign from the lower limit of integration. We implement cutoffs at both large and small  $k$ , for the ultraviolet and infrared divergences that arise in both the scalar and magnetic-field spectra. The cross correlations for  $n = 0$  and  $n = 2$  are

$$\begin{aligned} \langle (\delta\phi)B^2 \rangle &\simeq \frac{M}{16\pi^4 a^4(\eta_I)} \left( \frac{H_I}{M} \right)^2 \\ &\times \begin{cases} k_{\max}^4 (\ln r - \frac{25}{12}) & n = 0 \\ \eta_I^{-4} (100 + 24\ln^3 r - 72\ln^2 r \ln(-k_{\max} \eta_I)) & n = 2, \end{cases} \end{aligned} \quad (52)$$

where  $r = k_{\max}/k_{\min}$ , and  $k_{\max}$  and  $k_{\min}$  are upper and lower bounds on the run of wave vectors. In practice, we expect to link the minimum wave vector with the Hubble scale,  $k_{\min} \simeq 2\pi H_0$ , and the maximum wave vector with some galactic scale,  $k_{\max} \simeq 2\pi/\lambda$  where  $\lambda \sim \text{kpc}$ . Since  $|k\eta_I| \ll 1$ , we have discarded subdominant terms from the above results. The dimensionless cross-correlation coefficient  $X_{\delta\phi B^2}$ , formed from the ratio of the cross correlation with the root-mean-square amplitudes of the scalar and magnetic fields, gives

$$X_{\delta\phi B^2} \equiv \frac{\langle \delta\phi B^2 \rangle}{(\delta\phi)_{\text{rms}} B_{\text{rms}}^2} \simeq \begin{cases} \frac{1}{\pi} \left( \frac{H_I}{M} \right) \left( \ln r - \frac{25}{12} \right) / \sqrt{\ln r} & n = 0 \\ \frac{4}{9\pi} \left( \frac{H_I}{M} \right) (25 + 6\ln^3 r - 18\ln^2 r \ln(-k_{\text{max}} \eta_I)) / \sqrt{\ln^3 r} & n = 2. \end{cases} \quad (53)$$

If we consider a sufficiently wide range of scales, e.g.  $r \geq 10^4$ , then  $X(n=0) \simeq (H_I/M) \sqrt{\ln r}/\pi$  and  $X(n=2) \simeq 8(H_I/M) \sqrt{\ln r} \ln(-k_{\text{max}} \eta_I)/\pi$ . Using  $-k_{\text{max}} \eta_I \sim 10^{-27}$ , the cross-correlation coefficient in the presence of the amplification mechanism is enhanced by a factor of  $\sim 500$  over the case without the magnetic-field amplification mechanism. When the full range of inflationary length scales is taken,  $r \sim 10^{27}$ , then  $X(n=2) \simeq 2 \times 10^3 (H_I/M)$ . Since the cross-correlation coefficient cannot exceed unity, we infer an upper bound of  $H_I/M \leq 5 \times 10^{-4}$  which is consistent with naive expectations based on an inflationary scenario.

### C. The behavior in Fourier space

We now evaluate the triangle-shape dependence of the full three-point correlation function in Fourier space. To do so, we evaluate a ratio of the form

$$\frac{P_3(k_1, k_2, k_3)}{\sqrt{P_{\delta\phi}(k_1) P_B(k_2) P_B(k_3)}}, \quad (54)$$

to normalize the cross-correlation power spectrum. However, since this ratio is not dimensionless, given our Fourier conventions, we go to a discretized Fourier transform,

$$\int \frac{d^3 k}{(2\pi)^3} \rightarrow \frac{1}{V} \sum_{\vec{n}}, \quad (55)$$

and likewise, replacing the Dirac delta function with a Kronecker delta,

$$(2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) \rightarrow V \delta_{\vec{n}_1, \vec{n}_2}. \quad (56)$$

We presume a maximum length,  $L$ , so that the volume is  $V = L^3$  and mode numbers are  $k_i = 2\pi n_i/L$ . The scalar-field and magnetic-field power spectra are now

$$\langle (\delta\phi/M)^2 \rangle = \sum_{\vec{n}} e^{i\vec{n} \cdot (\vec{x} - \vec{y})/L} \tilde{P}_{\delta\phi}, \quad (57)$$

$$\tilde{P}_{\delta\phi} = V^{-1} P_{\delta\phi}/M^2, \quad (58)$$

$$\langle B^2 \rangle = \sum_{\vec{n}} e^{i\vec{n} \cdot (\vec{x} - \vec{y})/L} \delta_{\vec{n}_1, \vec{n}_2} \tilde{P}_B, \quad (59)$$

$$\tilde{P}_B = V^{-1} P_B, \quad (60)$$

so that  $\tilde{P}_{\delta\phi}$  is dimensionless and  $\tilde{P}_B$  has units of (energy)<sup>4</sup>. The three-point function becomes

$$\left\langle \frac{\delta\phi}{M} B^2 \right\rangle = \sum_{\vec{n}_1 + \vec{n}_2 + \vec{n}_3 = 0} e^{i(\vec{n}_1 \cdot \vec{x} + \vec{n}_2 \cdot \vec{y} + \vec{n}_3 \cdot \vec{z})/L} \tilde{P}_3, \quad \tilde{P}_3 = V^{-2} P_3, \quad (61)$$

where  $\tilde{P}_3$  has units of (energy)<sup>4</sup>. We can now build a dimensionless cross-correlation coefficient,

$$C_n = \frac{\tilde{P}_3(n_1, n_2, n_3)}{\sqrt{\tilde{P}_{\delta\phi}(n_1) \tilde{P}_B(n_2) \tilde{P}_B(n_3)}}, \quad (62)$$

where  $n_i$  for  $i = 1, 2, 3$  are the magnitudes of vectors  $\vec{n}_i$  that form a closed triangle.

For isosceles triangles with  $n_2 = n_3$ , the correlation  $C_n$  obtained for the case  $n = 0$  and  $n = 1$  is

$$C_0 = \frac{1}{8\pi^{3/2}} \frac{H_I}{M} \frac{(n_1 + n_2)(n_1 - 2n_2)^2}{n_1^{3/2} n_2^3},$$

$$C_1 = \frac{1}{16\pi^{3/2}} \frac{H_I}{M} \frac{N}{(n_1 + 2n_2)n_1^{3/2} n_2^5},$$

$$N = n_1^6 + 2n_1^5 n_2 - 2n_1^4 n_2^2 - 6n_1^3 n_2^3 + 4n_1^2 n_2^4 + 8n_1 n_2^5 + 16n_2^6, \quad (63)$$

where  $-1 \leq \cos\theta = \frac{1}{2} \frac{n_1^2}{n_2^2} - 1 \leq 1$ . An expression for  $C_2$  is easily calculated, but the result is rather long and unenlightening. The behavior of  $C_n(\cos\theta)$  for  $n = 0, 1, 2$ , and  $-2$  is illustrated in Fig. 2.

We find that there are two interesting limits for isosceles triangles with  $n_2 = n_3$ , first a squeezed triangle, with  $1 \leq n_1 \ll n_2$  or  $\theta = \pi$ , and second a flattened triangle, with

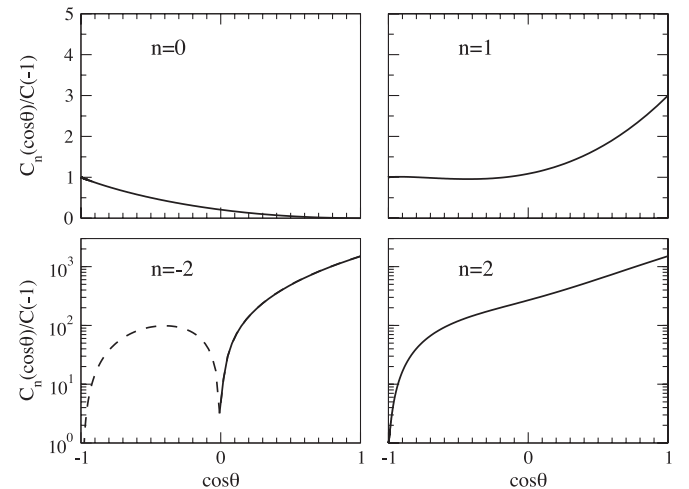


FIG. 2. The ratio  $C_n(\cos\theta)/C_n(-1)$  is shown for  $n = 0, 1, 2$ , and  $-2$ , as functions of  $\cos\theta$ . In the  $n = -2$  panel, the dashed line indicates where the absolute value has been taken. In the  $n = 2, -2$  cases we have used  $2\pi n_1 |\eta_I|/L \sim 10^{-27}$ , corresponding approximately to a Gpc length scale. Note that the case of cosmological interest is  $n = 2$ .

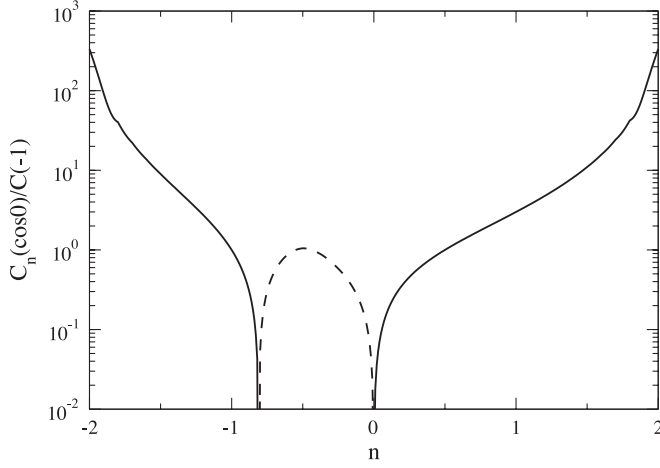


FIG. 3. The ratio  $C_n(1)/C(-1)$ , the ratio of the discretized Fourier-space cross-correlation coefficients for the flattened triangle to that of the universal result for the squeezed triangle, is shown as a function of  $n$ . No amplification,  $n = 0$ , yields zero cross correlation. Hence, the flattened triangle may be used as an indicator of an amplification mechanism. The ratio is negative along the dashed line, where we have taken the absolute value. We have set  $2\pi n_1|\eta_I/L| \sim 10^{-6}$  for ease of numerical computation; using  $2\pi n_1|\eta_I/L| \sim 10^{-27}$  to represent Gpc scales boosts the curve up to  $10^3$  near  $n = \pm 2$ . Note that the case of cosmological interest is  $n = 2$ .

$n_2 = n_1/2$  or  $\theta = 0$ . For the squeezed triangle we find the universal result

$$C_n(\cos \pi) = \frac{\sqrt{2}}{(2\pi n_1)^{3/2}} \frac{H_I}{M}, \quad (64)$$

for all values of  $n$ , as borne out by numerical integration for noninteger  $n$ . We suspect that this triangle configuration, with small  $n_1$  and large  $n_2, n_3$ , dominates the integration in Eq. (53), as a way to help explain the similarities seen in the real-space cross-correlation coefficients for different values of the index  $n$ .

The result, Eq. (64), suggests a natural reference point, so that a general expression for the discretized Fourier-space dependence of the cross correlation is

$$C_n(\cos \theta)/C_n(\cos \pi) = \frac{\pi}{4\epsilon n_1} \frac{2\cos \theta I_1 + (1 + \cos^2 \theta) I_2}{|H_{1/2+n}^{(1)}(\epsilon n_2) H_{1/2+n}^{(1)}(\epsilon n_3)|}, \quad (65)$$

where  $\epsilon \equiv -2\pi\eta_I/L \ll 1$ .

For a flattened triangle, we have  $C_0(\cos 0)/C_n(\cos \pi) = 0$ ,  $C_1(\cos 0)/C_n(\cos \pi) = 3$ , and  $C_{-2}(\cos 0)/C_n(\cos \pi) = 12(2 - \gamma - \ln(2\epsilon))$ , where  $\gamma$  is the Euler-Mascheroni constant. Note that the cross correlation vanishes for the unamplified case ( $n = 0$ ), but grows large for  $n = -2$ , where the argument of the log is  $\sim 10^{-27}$  for modes entering the

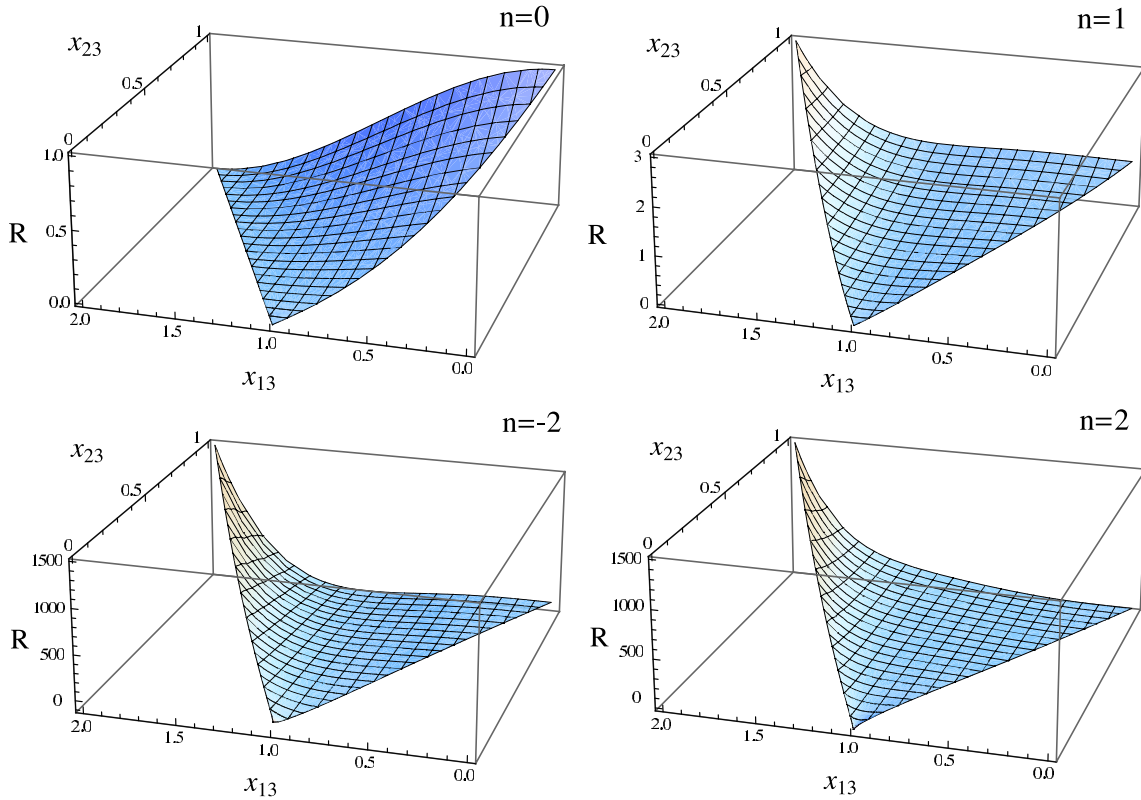


FIG. 4 (color online). The quantity  $R$ , defined in the text as the ratio of the Fourier-space cross-correlation coefficient to that of the universal result for the squeezed triangle, times a factor  $x_{23}^2$ , is shown as a function of the triangle side lengths. We have set  $2\pi n_3|\eta_I/L| \sim 10^{-27}$  for the cases  $n = \pm 2$ . Note that the case of cosmological interest is  $n = 2$ .



horizon today. The behavior of  $C_n(\cos 0)/C_n(\cos \pi)$  as a function of  $n$  is shown in Fig. 3.

To show the full Fourier-space triangle dependence of the cross correlation, we define the quantity

$$R \equiv \left(\frac{n_2}{n_3}\right)^2 \frac{C_n(\cos \theta)}{C_n(\cos \pi)} \quad (66)$$

and introduce the variables  $x_{23} \equiv n_2/n_3$  and  $x_{13} \equiv n_1/n_3$ , where  $0 \leq x_{23} \leq 1$  and  $1 - x_{23} \leq x_{13} \leq 1 + x_{23}$  cover the full set of triangles. The behavior for the cases  $n = 0, 1, 2$ , and  $-2$  are shown in Fig. 4. The figure helps illustrate the difference between the amplified and unamplified ( $n = 0$ ) cases, and also shows that the maximum value of  $R$  in the amplified case occurs for the flattened triangles, corresponding to the line  $x_{13} = 1 + x_{23}$ , along which  $\theta = 0$ . Squeezed triangles, where  $\theta = \pi$ , are located along  $x_{13} = 1 - x_{23}$ .

The amplitude of the zero-lag, real-space cross correlation is dominated by the cutoffs to the range of Fourier modes. At a finite Fourier mode, however, the flattened isosceles triangle produces the largest cross-correlation amplitude and hence the best opportunity to observe the signature of the mechanism responsible for amplifying the primordial magnetic field.

## V. CONCLUSIONS

We have considered a toy model in which a scalar field is coupled to electromagnetism in a fixed de Sitter background. The homogeneous time evolution of the scalar field breaks the conformal invariance of electromagnetism, resulting in quantum production of magnetic fields in addition to quantum production of scalar-field fluctuations. We then calculated the cross correlation between the scalar-field and the magnetic-field energy density. The dimensionless cross-correlation coefficient is proportional to the ratio  $H_I/M$ , which must be small if the effect of the scalar-field perturbation on the electromagnetic part of the Lagrangian can be considered small. However, this small quantity may be multiplied by a numerically large ( $\lesssim 500$ ) coefficient, suggesting a possibly strong (even order-unity) cross correlation.

We also studied the full triangle-shape dependence of the three-point correlation function in Fourier space. We found that it is nonzero for squeezed triangles (wherein the short Fourier component is that associated with the scalar-field mode), but may be considerably larger for flattened triangles (where the long Fourier mode is associated with the scalar field and twice as long as those associated with the magnetic field). These shape

dependences may be useful if such correlations are to be sought in the data.

Although we treat it as a toy model, our calculation provides the correlation between the curvaton and magnetic fields if the scalar field is identified as the curvaton. If primordial perturbations are further due to curvaton fluctuations, then the scalar-field–magnetic-field cross correlations derived here describe density-perturbation–magnetic-field correlations in the Universe today. If the scalar field is the inflaton, then there are additional steps to relate the scalar-field perturbation to the density-perturbation amplitude in the Universe today [30].

The cross correlation between primordial-seeded density perturbations and magnetic fields may be amenable to detection through the CMB. Cosmic magnetic fields present during the recombination era contribute to the CMB temperature and polarization signals. (See Ref. [15] for a detailed study.) Magnetic fields along the line of sight further distort the CMB by converting  $E$ -mode polarization into  $B$ -mode polarization, through Faraday rotation [16–20]. Primordial magnetic fields may also leave a non-Gaussian imprint on the statistics of the anisotropy pattern [21–25]. Current observations set the upper bound on a primordial magnetic field at the nG level [26–28]. (We also note that there have been claims of a lower bound on an extragalactic field [31].)

The correlation may also be accessible through a combined survey of large-scale structure and Faraday rotation [32]. The proposed Square Kilometer Array telescope, which is projected to be sensitive to variations of 0.1 nG across 100 Mpc, and the Low Frequency Array, which aims to explore the nG fields in intergalactic media [33–36], may offer more direct means to probe for the cross correlation. The detectability of the effect studied here will be the subject of future work [30].

## ACKNOWLEDGMENTS

M. K. thanks the Miller Institute for Basic Research in Science for support and the Department of Physics at the University of California, Berkeley, where part of this work was completed, for hospitality. R. C. and L. M. thank Caltech for hospitality, where part of this work was completed. This work was supported in part by NSF Grant No. AST-0349213 at Dartmouth College and by DOE Grant No. DE-FG03-92-ER40701, NASA Grant No. NNX10AD04G, and the Gordon and Betty Moore Foundation at Caltech.

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